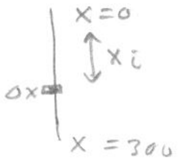
5. (12 points) A large crane is 300 feet above the ground. It has a cable with density 5 lbs/ft that reaches all the way down to the ground.

(a) If a 100 lbs object is attached to the bottom of the cable, how much total work is done in lifting the object the entire 300 feet?



LIFT OBJECT $= \underbrace{100}_{\text{CONST. FORCE}} \cdot \underbrace{300}_{\text{DIST}} = 30000$ ft-lbs

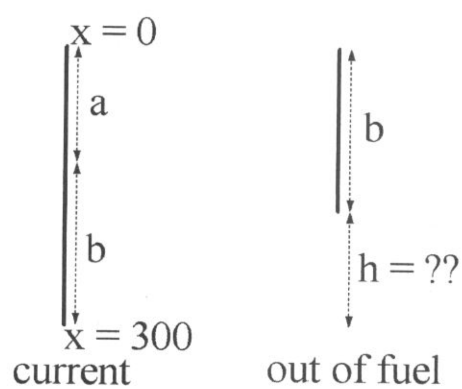
LIFT CABLE $\approx \sum_{i=1}^{n} 5\Delta x \, x_i$

$= \int_0^{300} 5x\,dx = \frac{5}{2}x^2 \Big|_0^{300} = \frac{5}{2}\, 90000 = 225{,}000$ ft-lbs

TOTAL $= 30{,}000 + 225{,}000 = \boxed{255{,}000 \text{ ft-lbs}}$

(b) At the end of the day, there is only enough fuel left for the crane to do 20,000 ft-lbs of work. Currently, the cable is extended the full 300 feet to the ground and there is NO object attached to the end. How high can it lift the cable before it runs out of fuel?

A picture is provided with labels you should find helpful, I suggest you find a first. (Give your final answer for h as a decimal to four digits).



WORK TO LIFT
$x=0$ TO $x=a$
SEGMENT $= \int_0^a 5x\,dx$

$= \frac{5}{2}x^2 \Big|_0^a = \frac{5}{2}a^2$

WORK TO LIFT
$x=a$ to $x=300$
SEGMENT $= \underbrace{5b}_{\text{CONST. FORCE}} \cdot \underbrace{a}_{\text{DIST}} = 5ab$

WANT: $\frac{5}{2}a^2 + 5ab = 20{,}000$

$b = 300 - a \Rightarrow \frac{5}{2}a^2 + 5a(300-a) = 20000$

$\frac{5}{2}a^2 - 5a^2 + 1500a - 20000 = 0$

$-\frac{5}{2}a^2 + 1500a - 20000 = 0$

$-5a^2 + 3000a - 40000 = 0$

$a = \dfrac{-3000 \pm \sqrt{3000^2 - 4(-5)(-40000)}}{2(-5)}$

$a = \dfrac{-3000 \pm \sqrt{8200000}}{-10}$

$a = 13.64357$ or $\cancel{a = 586.3564}$

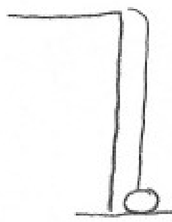
SAME

Thus, $b = 300 - a = 286.35642$

$\boxed{h = 13.6436 \text{ ft}}$

6. (8 points) A bag of sand is lifted from the ground to the top of a 30 foot high building at a constant speed with a cable that weighs 2 lb/ft. A small tear in the bag causes sand to slowly pour out. Initially the bag contains 100 pounds of sand, but the sand leaks out at a constant rate and the bag weighs 90 pounds just as it reaches the 30 foot height. How much work is done?
(Hint: Find a linear equation for the force (weight) of the bag of sand at a given height.)

work = work to lift cable
+ work to lift sandbag



work to lift cable

$$= \int_0^{30} 2x\,dx = x^2 \Big|_0^{30} = \boxed{900}$$

work to lift sandbag

$$= \int_0^{30} \text{force}\,dx$$

$x = 0$ force $= 90$

$x = 30$ force $= 100$

force $= mx + b$

$$m = \frac{100-90}{30-0} = \frac{1}{3}$$

$b = 90$

force $= \frac{1}{3}x + 90$

$$= \int_0^{30} \frac{1}{3}x + 90\,dx = \frac{1}{6}x^2 + 90x \Big|_0^{30}$$

$$= \frac{1}{6}900 + 2700 = \boxed{2850}$$

$$\text{Work} = 900 + 2850 = \boxed{3750 \text{ ft-lbs}}$$

Note: you could also take $x=0$ at the bottom in which case the force equation becomes force $= -\frac{1}{3}x + 100$. (But you get the same value)

5. (8 pts) After Dr. Loveless dries off, he continues his work out. He starts to lift a sandbag. The sandbag weighs 50 pounds when it is on the ground. As he lifts the bag it leaks out sand at a constant linear rate. When the sandbag is lifted 2 feet, it weighs 46 pounds. Before he passes out, Dr. Loveless does 145 foot-pounds of work in lifting the sandbag. How high did he lift the sandbag?
(Hint: Start by finding the linear function for weight (force) in terms of height.)

$f(x) = mx + b \qquad m = \frac{46-50}{2-0} = -2$

$\boxed{f(x) = -2x + 50 \text{ pounds}}$ $\quad x =$ dist. from bottom.

$x = 2$ 46 lbs

$x = 0$ 50 lbs

only makes sense up to $x = 25$

$\text{WORK} = \int_0^a -2x + 50\,dx \overset{?}{=} 145 \text{ ft-lbs}$. FIND $a = ?$

$-x^2 + 50x \Big|_0^a$

$-a^2 + 50a \overset{?}{=} 145$

$0 = a^2 - 50a + 145$

$a = \frac{50 \pm \sqrt{50^2 - 4 \cdot 145}}{2}$

the other sol'n is 46.9069023

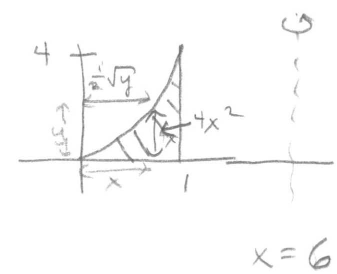
take the answer less than 25

$\boxed{a = \frac{50 - \sqrt{50^2 - 4 \cdot 145}}{2} = \frac{50 - \sqrt{1920}}{2} \approx 3.091977 \text{ feet}}$

6. (6 points) Consider the region, R, bounded by $y = 4x^2$ and the x-axis between $x = 0$ and $x = 1$. Using both methods, cylindrical shells and cross-sectional slicing, set up two integrals for the volume of the solid obtained by rotating the region R about the **vertical line** $x = 6$.
Only set up, DO NOT EVALUATE.

(a) Cross-sectional slicing:

$\boxed{\int_0^4 \pi\left(6 - \tfrac{1}{2}\sqrt{y}\right)^2 - \pi\, 5^2\, dy}$

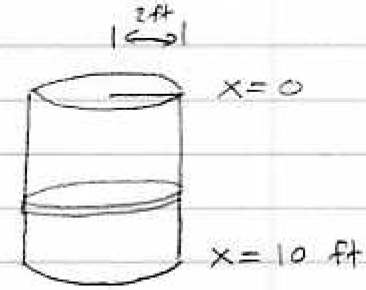


$= 14\pi \approx 43.9823$

(b) Cylindrical Shells:

$\boxed{\int_0^1 2\pi(6 - x)\, 4x^2\, dx}$

⑤ (a) Let x = dist. from top.



Force of a horiz. slice $= 62.5(\text{Area})\,\Delta x$

$= 62.5\,\pi r^2\,\Delta x$

$= 62.5\pi\, 2^2\,\Delta x$

Dist $= x$

$$\text{WORK} = \underbrace{\int_5^{10}}_{\text{HALF FULL}} \underbrace{x}_{\text{DIST}} \underbrace{62.5\pi\, 2^2\, dx}_{\text{FORCE}}$$

$= 250\pi \int_5^{10} x\,dx$

$= 250\pi\ \frac{1}{2}x^2\Big|_5^{10} = 125\pi\left[10^2 - 5^2\right]$

$= 125\pi(75) = \boxed{9375\pi \text{ ft-lbs}}$

$\approx \boxed{29452.4311274 \text{ ft-lbs}}$

(b)

$\text{WORK} = \int_0^8 F(x)\,dx$

$= \int_0^8 40 + 50e^{-x/2}\,dx$

$= 40x + 50(-2)e^{-x/2}\Big|_0^8$

$= (40(8) - 100e^{-8/2}) - (40(0) - 100e^{-0/2})$

$= 320 - 100e^{-4} + 100$

$= 420 - 100e^{-4}$ Joules

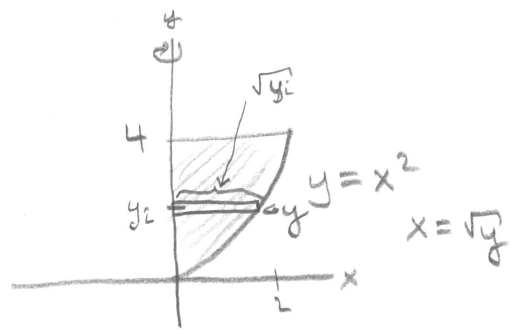
$= 420 - \frac{100}{e^4}$

$\approx \boxed{418.1684361 \text{ Joules}}$

5. (10 points) Consider the region R in the first quadrant of the xy-plane bounded by $y = x^2$, $y = 4$ and the y-axis ~~for some positive number a.~~ The water in a full tank is in the shape of the solid obtained by rotating R about the y-axis.

Assume all lengths are in meters, so the tank is 4 meters high. And remember the density of water is 1000 kg/m^3 and gravity is 9.8 m/s^2.

Set up and evaluate an integral for the work required to pump all the water to the top of the tank and over the edge.



WORK ON A SLICE $= \underbrace{\underbrace{1000 \cdot 9.8}_{N/m^3} \cdot \underbrace{\pi(\sqrt{y_i})^2 \Delta y}_{m^3}}_{\text{FORCE}} \cdot \underbrace{(4 - y_i)}_{\substack{m \\ \text{DIST}}}$

$$\text{WORK} = \int_0^4 9800\,\pi(\sqrt{y})^2(4-y)\,dy$$

$$= 9800\pi \int_0^4 4y - y^2\,dy$$

$$= 9800\pi \left[2y^2 - \tfrac{1}{3}y^3 \Big|_0^4\right]$$

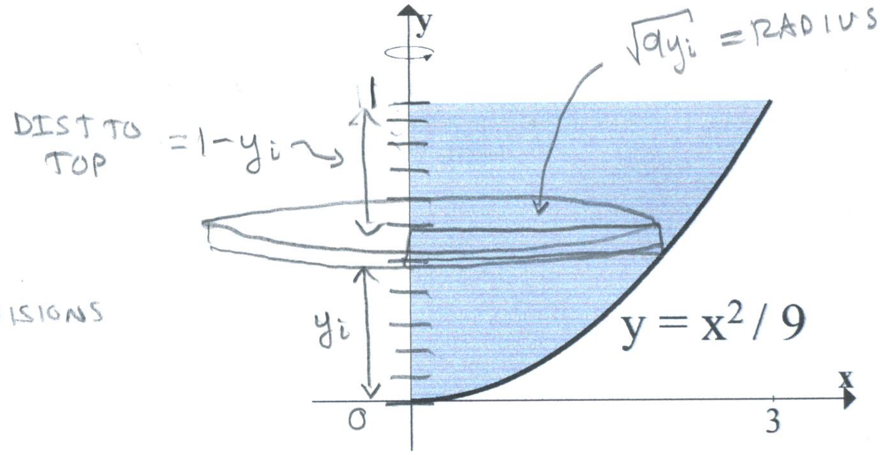
$$= 9800\pi \left[\left(2 \cdot 4^2 - \tfrac{1}{3}4^3\right) - (0)\right]$$

$$= 9800\pi \left[32 - \tfrac{64}{3}\right]$$

$$= 9800\pi \left[\tfrac{96}{3} - \tfrac{64}{3}\right] = \boxed{\tfrac{9800\pi}{3} \cdot 32 \quad J}$$

5. (10 points) The portion of the graph $y = \frac{1}{9}x^2$ between $x = 0$ and $x = 3$ is rotated around the y-axis to form a container. The container is full of a liquid that has density 100 lbs/ft^3. Find the work required to pump all of the liquid out over the side of the container. (Distance is measured in feet).

$$y = \frac{1}{9}x^2$$
$$\Rightarrow 9y = x^2$$
$$\Rightarrow x = \sqrt{9y}$$



BREAK INTO n SUBDIVISIONS

$$\Delta y = \frac{1-0}{n} = \frac{1}{n}$$
$$y_i = \frac{i}{n}$$

FOR EACH SUBDIVISION, WE APPROXIMATE

$$\text{FORCE} \approx \left(100 \tfrac{\text{lbs}}{\text{ft}^3}\right)\underbrace{\Big(\underbrace{\pi\left(\sqrt{9y_i}\right)^2}_{\text{AREA}}\ \underbrace{\Delta y}_{\text{THICKNESS}}\ \text{ft}^3\Big)}_{\text{VOLUME}} = 100\pi\, 9y_i\, \Delta y \ \text{lbs}$$

$$\text{DIST TO TOP} \approx 1 - y_i$$

$$\text{TOTAL WORK} = \lim_{n\to\infty} \sum_{i=1}^{n} 100\pi\, 9y_i\, \Delta y\, (1-y_i)$$
$$= \int_0^1 100\pi\, 9y\,(1-y)\,dy$$
$$= 900\pi \int_0^1 y - y^2\,dy$$
$$= 900\pi\left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big|_0^1\right)$$
$$= 900\pi\left(\tfrac{1}{2} - \tfrac{1}{3}\right)$$
$$= 900\pi\,\tfrac{1}{6} = \boxed{150\pi \ \text{ft-lbs}}$$

5. (10 points) You run out of water balloons. So you devise a scheme to dump a bucket of water on your instructor's head instead. Here is your plan:

(a) A tank full of rainwater is outside your dorm. The shape of the tank is described as follows: Consider the region R in the first quadrant of the xy-plane bounded by $y = x^2$, $y = 1$ and the y-axis (lengths are in meters). The full tank is in the shape of the solid obtained by rotating R about the y-axis.
You plan to pump all the water to the top of the tank and over the edge into your bucket.

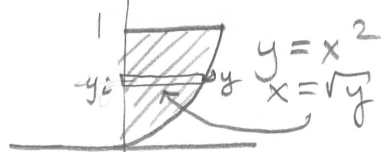
(b) Once all the water is in your bucket. The bucket is lifted by cable to the roof of your dorm (where you will wait for your instructor to walk by). The cable weighs 5 Newtons per meter and the *empty* bucket weighs 100 Newtons. The top of the building is 20 meters high.

Recall the density of water is 1000 kg/m³ and gravity is 9.8 m/s².

Find the total ~~about~~ amount of work done in pumping out the water and lifting the full bucket to the roof of your dorm. (Give your final answer as a decimal in Joules).

PUMPING WATER

$$\sum_{i=1}^{n} \underbrace{\underbrace{1000\cdot 9.8}_{N/m^3}\cdot \underbrace{\pi(\sqrt{y_i})^2\Delta y}_{m^3}}_{\text{FORCE FOR A SLICE}}\ \underbrace{\underbrace{(1-y_i)}_{m}}_{\text{DIST FOR A SLICE}}$$

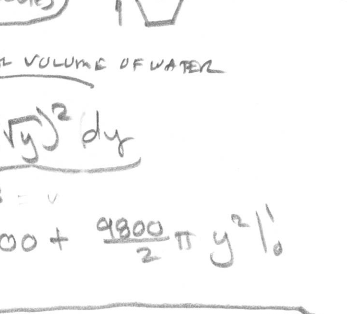


WORK TO PUMP WATER

$$= \int_0^1 9800\pi(\sqrt{y})^2(1-y)\,dy = 9800\pi\int_0^1 y(1-y)\,dy$$
$$= 9800\pi\int_0^1 y - y^2\,dy = 9800\pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big|_0^1\right]$$
$$= 9800\pi\left[\left(\tfrac{1}{2}-\tfrac{1}{3}\right) - 0\right] = \frac{9800}{6}\pi = \frac{4900}{3}\pi \text{ Joules}$$

LIFTING CABLE

$$\sum_{i=1}^{n} \underbrace{\underbrace{5}_{N/m}\ \underbrace{\Delta t}_{m}}_{\text{FORCE FOR A SLICE}}\ \underbrace{\underbrace{t_i}_{m}}_{\text{DIST FOR A SLICE}}$$



WORK TO LIFT CABLE ALONE

$$= \int_0^{20} 5t\,dt = \frac{5}{2}t^2\Big|_0^{20} = \frac{5}{2}400 = 1000 \text{ Joules}$$

LIFTING BUCKET

$$\text{TOTAL WEIGHT} = \underbrace{100}_{\text{EMPTY WEIGHT OF BUCKET}} + \underbrace{1000\cdot 9.8}_{N/m^3}\ \underbrace{\overbrace{\int_0^1 \pi(\sqrt{y})^2\,dy}^{\text{TOTAL VOLUME OF WATER}}}_{m^3}$$
$$= 100 + 9800\pi\int_0^1 y\,dy = 100 + \frac{9800}{2}\pi\, y^2\Big|_0^1$$
$$= 100 + 4900\pi \ N$$

$$\text{WORK} = \text{FORCE} * \text{DIST} = \underbrace{(100 + 4900\pi)}_{N}\underbrace{(20)}_{m} = 2000 + 98000\pi \text{ J}$$

$$\text{TOTAL} = \frac{4900}{3}\pi + 1000 + 2000 + 98000\pi \approx \boxed{316{,}007.35 \text{ J}}$$